Characteristic function for the stationary state of one-dimensional dynamic system with Lévy noise

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Abstract

A practical method is developed for calculating the characteristic function of diffusions driven by Lévy white noise. The method is based on Itô's formula for semimartingales, a differential equation developed for the characteristic function of diffusions driven by Poisson white noise with jumps that may not have finite moments, and approximate representations of the Lévy white noise process. Numerical results show that the proposed method is very accurate and is consistent with previous theoretical findings.

1 Introduction

The Fokker-Planck equation is a partial differential equation for the density of the state of a dynamic system driven by Gaussian white noise involving temporal and spatial derivatives of orders 1 and orders 1 and 2, respectively. Extensions of the Fokker-Planck equation to solve dynamic systems driven by non-Gaussian white noise are less simple. For example, the Fokker-Planck equation for the density of the state of a dynamic system subjected to Poisson white noise has an infinite number of terms consisting of spatial derivatives of orders from 1 to infinity, and is defined if the jumps of the Poisson white noise process have moments of any order ([7], Section 7.3.1.3, [8]). If the driving white noise corresponds to a symmetric α-stable process, the density of the system state satisfies a partial differential equation similar to the classical Fokker-Planck equation except that the term involving the second order spatial derivative of the state density in the classical equation is replaced by a term consisting of a fractional spatial derivative of the state density of order α [2, 5]. In addition to complexity, it may be an additional difficulty with the use of extended versions of the classical Fokker-Planck equation related. The use of these equations is only possible for diffusions driven by Poisson and/or Lévy white noise that have a density. We avoid this difficulties by dealing with characteristic functions of diffusion processes. In our discussion, the Gaussian, Poisson, and α-stable white noise processes are interpreted as formal temporal derivatives of Brownian, compound Poisson, and symmetric α-stable processes, respectively.

Our main objective is the development of a practical method for calculating the characteristic function of diffusions driven by a class of Lévy white noise processes, defined as the formal derivative of symmetric α-stable Lévy motion process L_α. The method is based on (1) Itô's formula for semimartingales, (2) a differential equation developed for the characteristic function of diffusions driven by Poisson white noise with jumps that may not have finite moments, and (3) approximate representations of L_α consisting of the large jumps of L_α or the large jumps of L_α superposed on scaled Brownian motions.

It is shown that (1) the solution of the differential equation for the characteristic function requires, in addition to boundary conditions following from the defining properties of the characteristic function, some other boundary conditions of the type proposed in [5], (2) differential equations for the characteristic function of diffusions driven by Gaussian and Lévy white noise can be obtained as limits of differential equations for the characteristic functions of diffusions driven by Poisson white noise processes, (3) the differential equation for the characteristic function of diffusions driven by Lévy white noise obtained in the paper coincides with that in [2], and (4) the proposed approximate method for analyzing diffusions driven by Lévy white noise is highly accurate and is consistent with previous theoretical findings [11].

2 Itô's formula for arbitrary diffusions

Let $X(t)$, $t \geq 0$, be a real-valued semimartingale and considerer the mapping $X(t) \mapsto \exp\big(i\,u\,X(t)\big)$ for some $u \in \mathbb{R}$. Since the second derivative of $\exp\big(i\,u\,X(t)\big)$ with respect to $X(t)$ is continuous, Itô's lemma can be applied to this function in a time interval $[t, t']$, $0 \leq t < t'$, and gives ([7], Section 4.6.1)

$$e^{i\,u\,X(t')} - e^{i\,u\,X(t)} = \int_{t+}^{t'} i\,u\,e^{i\,u\,X(s-)}\,dX(s) + \frac{1}{2}\int_{t+}^{t'} (i\,u)^2\,e^{i\,u\,X(s-)}\,d[X,X]^c(s) + \sum_{t<s\leq t'} \Big(e^{i\,u\,X(s)} - e^{i\,u\,X(s-)} - i\,u\,e^{i\,u\,X(s-)}\,\Delta X(s)\Big), \tag{1}$$

where $\Delta X(s) = X(s) - X(s-)$ denotes the jump of X at time s and $[X,X]^c$ is the continuous part of the quadratic variation process $[X,X]$.

We use Eq. 1 in the following sections to derive equations for the characteristic function $\varphi(u;t) = E\big[e^{i\,u\,X(t)}\big]$ of $X(t)$ defined by

$$dX(t) = -X(t-)^{2\,m+1}\,dt + dV(t), \quad t \geq 0, \tag{2}$$

where V denotes the driving process and $m \geq 0$ is an integer. In the following sections V is a compound Poisson process, a symmetric α-stable Lévy motion process L_α, a Brownian motion, a sum of a Brownian motion and a compound Poisson process. We note that Eq. 2 has a strongly unique solution since its drift function $a(x) = -x^{2\,m+1}$ is Lipschitz on compact intervals, $a(0) = 0$, and $a(x)$ is a decreasing function [11]. The unique solution X of Eq. 2 is a semimartingale.

3 Poisson white noise

Let V be the compound Poisson process

$$C(t) = \sum_{k=1}^{N(t)} Y_k, \quad t \geq 0, \tag{3}$$

where N denotes a homogeneous Poisson counting process with intensity $\lambda > 0$ and $\{Y_k\}$ are independent identically distributed random variables that are unrelated to N. The jumps $\{Y_k\}$ of C may not have moments.

Let $n \geq 1$ be an integer, $Y_{n,k} = Y_k \, 1\big(|Y_k| \leq n\big)$, and

$$C_n(t) = \sum_{k=1}^{N(t)} Y_{n,k}, \quad t \geq 0, \tag{4}$$

be a compound Poisson process obtained from $C(t)$ by modifying its jumps. Denote by X_n the solution of Eq. 2 with $V = C_n$.

Let $\Delta t > 0$ be a small time increment, and denote by $\varphi_n(u;t) = E\big[e^{i\,u\,X_n(t)}\big]$ the characteristic function of $X_n(t)$. The expectation of Eq. 1 for $t' = t + \Delta t$ and X_n in Eq. 2 with $V = C_n$ is

$$\begin{aligned} \varphi_n(u; t+\Delta t) - \varphi_n(u;t) = & - i\,u\,E\left[\int_{t+}^{t+\Delta t} X_n(s-)^{2\,m+1}\, e^{i\,u\,X_n(s-)}\, ds\right] \\ & + E\left[\sum_{t<s\leq t+\Delta t} e^{i\,u\,X_n(s-)} \left(e^{i\,u\,\Delta X_n(s)} - 1\right)\right], \end{aligned} \tag{5}$$

since $[X_n, X_n]^c(s) = 0$ and X has a finite number of jumps in any bounded time interval a.s.

Suppose that the initial law of $X_n(0)$ in Eq. 2 is such that $E\big[\big|X_n(0)\big|^{2m+1}|\big] < \infty$. Since for all $t > 0$

$$\big|X_n(t)\big| \leq \big|X_n(0)\big| + \sum_{k=1}^{N(t)} \big|Y_{n,k}\big|,$$

we may use Fubini's theorem to interchange the order of expectation and integration in the first term on the right side of Eq. 5. Since $E\big[X_n(s-)^{2\,m+1}\, e^{i\,u\,X_n(s-)}\big]$ is a continuous function of s, the expectation of the first integral on the right side of Eq. 5 is $E\big[X_n(t)^{2\,m+1}\, e^{i\,u\,X_n(t)}\big]\,\Delta t + o(\Delta t)$. By the finiteness of the moment of order $(2m+1)$ of X, we have

$$E\big[X_n(t)^{2\,m+1}\, e^{i\,u\,X_n(t)}\big] = i^{-(2\,m+1)}\, \frac{\partial^{2\,m+1} \varphi_n(u;t)}{\partial u^{2\,m+1}}\,.$$

The second term on the right side of Eq. 5 is treated as follows. The jumps of X_n coincides with those of C_n, that is, $\Delta X_n(s) = \Delta C_n(s)$, and C_n has 0, 1, and two or more jumps in $(t, t+\Delta t]$ with probabilities $P\big(N(\Delta t) = 0\big) \simeq 1 - \lambda\,\Delta t$, $P\big(N(\Delta t) = 1\big) \simeq \lambda\,\Delta t$, and $P\big(N(\Delta t) \geq 2\big) \simeq o(\Delta t)$, respectively, under the assumption $\lambda\,\Delta t \ll 1$. We note that the sum under the second expectation on the right side of Eq. 5 is dominated by twice the number of the jumps N has in the interval $(t, t+\Delta t]$. This sum is equal to zero if N has no jump in $(t, t+\Delta t]$, that is, $N(\Delta t) = 0$, and its expectation on the event $\{N(\Delta t) > 1\}$ is $o(\Delta t)$. Therefore, we have

$$E\Big[\sum_{t<s\le t+\Delta t} e^{i\,u\,X_n(s-)}\left(e^{i\,u\,\Delta X_n(s)} - 1\right)\Big] = \lambda\,\Delta t\,\varphi_n(u;t)\left(E[e^{i\,u\,Y_{n,1}}] - 1\right) + o(\Delta t).$$

The above considerations show that Eq. 5 becomes

$$\varphi_n(u;t+\Delta t) - \varphi_n(u;t) = \left(-\,i\,u\,\frac{\partial^{2\,m+1}\varphi_n(u;t)}{\partial u^{2\,m+1}} + h_n(u)\,\varphi_n(u;t)\right)\Delta t + o(\Delta t), \qquad (6)$$

where $h_n(u) = \lambda\left(E\big[e^{i\,u\,Y_{n,1}}\big] - 1\right)$. We obtain

$$\frac{\partial\varphi_n(u;t)}{\partial t} = (-1)^{m+1}\,u\,\frac{\partial^{2\,m+1}\varphi_n(u;t)}{\partial u^{2\,m+1}} + h_n(u)\,\varphi_n(u;t), \quad n = 1, 2, \ldots, \qquad (7)$$

by dividing with Δt and taking the limit of the resulting expression as $\Delta t \to 0$.

Equation 2 admits a unique stationary solution, which has finite moments for $V = C_n$. This results for $m = 0$ directly since the solution can be written explicitly, and for $m > 0$ by the arguments in [11]. For the stationary solution $\varphi_n(u;t)$ is independent of t, and we denote it by $\varphi_{n,s}(u)$. This function satisfies the ordinary differential equation

$$(-1)^{m+1}\,u\,\frac{d^{2\,m+1}\varphi_{n,s}(u)}{du^{2\,m+1}} + h_n(u)\,\varphi_{n,s}(u) = 0, \quad n = 1, 2, \ldots. \qquad (8)$$

If the jumps of C_n are symmetric about 0, then $\varphi_{n,s}$ is a real-valued even function since, if X is a solution for Eq. 2 with $V = C$ so is $-X$ for $V = -C \stackrel{d}{=} C$. Hence, it is sufficient to solve Eq. 7 in the range $[0, \infty)$.

Consider now the solution X of Eq. 2 driven by the compound Poisson noise $V = C$ in Eq. 3. Let $\varphi(u;t) = E\big[e^{i\,u\,X(t)}\big]$ be the characteristic function of $X(t)$. It follows from Theorem 5.4 in [9] that if one chooses the initial laws of the sequence of solution $\{X_n, n \ge 1\}$ to Eq. 2 such that $\varphi_n(u;0) \to \varphi(u;0)$ for all real u as $n \to \infty$, then $\varphi_n(u;t) \to \varphi(u;t)$ also for all positive t. Noticing also that $h_n(u) \to h_n(u) = \lambda\left(E\big[e^{i\,u\,Y_1}\big] - 1\right)$ as $n \to \infty$, it is logical to expect that that the limiting characteristic φ is the solution of

$$\frac{\partial\varphi(u;t)}{\partial t} = (-1)^{m+1}\,u\,\frac{\partial^{2\,m+1}\varphi(u;t)}{\partial u^{2\,m+1}} + h(u)\,\varphi(u;t)\,. \qquad (9)$$

We also know by arguments outlined previously in this section that Eq. 2 with $V = C$ admits a unique stationary distribution. One expects that the characteristic function φ_s of this solution satisfy the ordinary differential equation

$$(-1)^{m+1}\,u\,\frac{d^{2\,m+1}\varphi_s(u)}{du^{2\,m+1}} + h(u)\,\varphi_s(u) = 0. \qquad (10)$$

Here is an argument that the Eq. 10 indeed holds for all $u \neq 0$. Using the tightness argument of Section 4 in [11] together with the uniqueness of the stationary distribution, we know that the stationary distribution of the solution of Eq. 2 with $V = C_n$ converges weakly to that of Eq. 2 with $V = C$. Therefore, $\varphi_{n,s}(u) \to \varphi_s(u)$ for all real u as $n \to \infty$. Consider an arbitrary interval (a_1, a_2), $0 < a_1 < a_2 < \infty$. Then the sequence of derivatives $d^{2m+1}\varphi_{n,s}(u)/du^{2m+1}$ converges uniformly on (a_1, a_2) by Eq. 8. Since $\varphi_{n,s}(u) \to \varphi_s(u)$, we conclude that $\varphi_s(u)$ is $2m+1$ times differentiable on (a_1, a_2) and the above sequence of derivatives converges to the corresponding derivative of $\varphi_s(u)$. Therefore, Eq. 10 holds on the interval (a_1, a_2), and because of the arbitrariness of the choice of this interval, the differential equation holds for all $u \neq 0$. We conjecture that the partial differential equation Eq. 9 holds as well.

The solutions of Eqs. 7 and 8 require initial and boundary conditions and only boundary conditions, respectively. The initial condition for Eq. 7 follows from the probability law of the initial state $X_n(0)$ and is $\varphi_n(u;0) = E\big[e^{i\,u\,X_n(0)}\big]$. The boundary conditions for Eqs. 21 and 8 can be divided in two groups. The boundary conditions in the first group follow from the definition of the characteristic function, and are (i) $\varphi_n(0;t) = 1$, $t \geq 0$, or $\varphi_{n,s}(0) = 1$ and (ii) $|\varphi_n(u;t)| \leq 1$ or $|\varphi_{n,s}(u)| \leq 1$ for all $t \geq 0$ and $u \in \mathbb{R}$. Unless $m = 0$ in Eq. 2, these boundary conditions are insufficient for solving Eqs. 7 and 8. The boundary conditions in the second group relate to properties of $X_n(t)$. For example, $\varphi_n(u;t)$ is differentiable at any order at $u = 0$ for any $t \geq 0$ since $X_n(t)$ has finite moments of any order.

Similar initial and boundary conditions can be imposed for Eqs. 9 and 10, except for the boundary conditions in the second group. These boundary conditions result from developments in [11]. Let $X(t)$ be the stationary state of Eq. 2 with $V = C$, T be the time of the last jump of C prior to t, and $U = t - T$. Then

$$|X(t)| = \frac{1}{\big(|X(T)|^{-2\,m} + 2\,m\,(t-T)\big)^{1/(2\,m)}} \leq \frac{1}{\big(2\,m\,U\big)^{1/(2\,m)}} \tag{11}$$

since X is the solution of $\dot{X} = -X^{2\,m+1}$ in the time interval $(T, t]$. Since U is an exponential variable with expectation $1/\lambda$, we have

$$P\big(|X(t)| > x\big) \leq P\big(U \leq 1/(2\,m\,x^{2\,m})\big) = 1 - \exp\big(-\frac{\lambda}{2\,m}\,x^{-2\,m}\big) \leq \frac{\lambda}{2\,m}\,x^{-2\,m}, \quad x > 0, \tag{12}$$

so that the upper tails of the stationary distribution of $X(t)$ decays at least as fast as $x^{-2\,m}$. Accordingly, the stationary state X has finite moments up to order $2\,m-1$ so that its characteristic function satisfies the conditions $|\varphi^{(k)}(0)| < \infty$, $k = 1, 2, \ldots, 2\,m-1$, where $\varphi^{(k)}(u) = d^k\varphi(u)/du^k$. If the the jumps of C are symmetric about zero, that is, $Y_1 \overset{d}{=} -Y_1$, then we also have $X(t) \overset{d}{=} -X(t)$ and

$$\varphi_s^{(1)}(0) = \varphi_s^{(3)}(0) = \cdots = \varphi_s^{(2\,m-1)}(0) = 0. \tag{13}$$

Analytical solutions for Eqs. 7, 8, 9, and 10 are possible only in special cases, but these equations can be solved numerically. For example, suppose that C has symmetric jumps and $m = 1$ in Eq. 2. Let $\bar{u} > 0$ be sufficiently large, $\Delta u = \bar{u}/n$, $u_r = r\,\Delta u$, $\varphi_r = \varphi_s(u_r)$, and

$$\xi_r = \frac{2\,\lambda\,(\Delta u)^2}{r}\left(E\big[e^{i\,u_r\,Y_1}\big] - 1\right). \tag{14}$$

The finite difference approximation of Eq. 10 is

$$-\varphi_{r-2} + 2\,\varphi_{r-1} + \xi_i\,\varphi_r - 2\,\varphi_{r+1} + \varphi_{r+2} = 0, \quad r = 1, \ldots, n, \tag{15}$$

and results by using central finite differences to approximate the third derivative of φ_s in Eq. 10. The boundary conditions attached to Eq. 15 are $\varphi_0 = 1$, $\varphi_1 = \varphi_{-1}$, and $\varphi_{n+k} = 0$, $k \geq 1$. The first condition is an essential property of the characteristic function. The second condition is valid since the characteristic function is a real-valued even function. It also results from the finite difference approximation of the condition $\varphi_s'(0) = 0$. The third boundary condition results from the fact that X has a density (Appendix 1), which implies $\varphi_s(u) \to 0$ as $u \to \infty$.

The following three subsections examine the cases in which the driving noise $V = C$ has symmetric α-stable jumps, $V = C$ has jumps in L_2 with mean 0 and variance decreasing with the intensity of the Poisson process N in the definition of C, and $V = \sigma\,B + C$, where σ is a constant and B denotes a standard Brownian motion process.

3.1 Poisson white noise with α-stable jumps

Suppose that $Y_1 \sim S_\alpha(1, 0, 0)$ is a symmetric α-stable variable with shape parameter α and scale 1. The arguments used to established Eqs. 7 and 8 remain valid provided that an appropriate definition is used for $h_n(u)$. Then the characteristic function φ_s of the stationary solution of Eq. 2 driven by a compound Poisson process with α-stable jumps is given by Eq. 10 with

$$h(u) = \lambda\,\big(\exp\big(-|u|^\alpha\big) - 1\big). \tag{16}$$

It is expected that the characteristic function φ of the transient solution of Eq. 2 satisfies Eq. 9 with $h(u)$ given by Eq. 16.

The resulting versions of Eqs. 9 and 10 can be solved numerically by, for example, the finite difference formulation in Eq. 15 with $E\big[e^{i\,u\,Y_1}\big] = \exp\big(-|u|^\alpha\big)$. The boundary conditions in Eq. 13 and prior to this equation can be used for solution.

3.2 Gaussian white noise

Let

$$C^{(n)}(t) = \sum_{k=1}^{N^{(n)}(t)} Y_k^{(n)}, \quad n = 1, 2, \ldots, \tag{17}$$

be a sequence of compound Poisson processes and let λ_n be the intensity of the Poisson counting process $N^{(n)}$. The characteristic functions of a standard Brownian motion $B(t)$ and of $C^{(n)}(t)$ in Eq. 17 are

$$\varphi_B(u; t) = E\left[e^{i\,u\,B(t)}\right] = e^{-u^2\,t/2} \tag{18}$$

and ([6], Section 3.3)

$$\varphi^{(n)}(u; t) = \exp\left\{\lambda_n\,t\,\left(E\left[e^{i\,u\,Y_1^{(n)}}\right] - 1\right)\right\}. \tag{19}$$

We first show that $\varphi^{(n)}$ in Eq. 19 converges to φ_B in Eq. 18 as $n \to \infty$, that is, $\lim_{n\to\infty} \varphi^{(n)}(u;t) = \varphi_B(u;t)$ for all $t \geq 0$ and $u \in \mathbb{R}$, under the assumptions: (1) the jumps $Y_k^{(n)}$ of $C^{(n)}$ are in L_2 such that $E[Y_k^{(n)}] = 0$ and $\lambda_n \, E\big[\big(Y_k^{(n)}\big)^2\big] = 1$ for all $n = 1, 2, \ldots$, (2) the family of random variables $\{Z_{n,k}^2\}$ is uniformly integrable, where $Z_{n,k} = \lambda_n^{1/2} \, Y_k^{(n)}$, and (3) $\lambda_n \to \infty$ as $n \to \infty$. Recall that a collection $\mathcal{U}$ of random variables is said to be uniformly integrable if $\sup_{U\in\mathcal{U}} E\big[|U| \, 1\big(|U| > \xi\big)\big] \to 0$ as $\xi \to \infty$.

Prior to proving this statement we note that (i) the driving noise $C^{(n)}$ has the same second-moment properties as the Brownian motion B for any n since $E[C^{(n)}(t)] = 0$ and $E[C^{(n)}(t) \, C^{(n)}(s)] = t \wedge s$, (ii) $Z_{n,k}$ is a square integrable random variable with $E[Z_{n,k}] = 0$ and $E[Z_{n,k}^2] = 1$ for all $n \geq 1$, (iii) the convergence $\varphi^{(n)} \to \varphi_B$ implies the convergence of the finite dimensional distributions of $C^{(n)}$ to those of B since these processes have stationary independent increments, and (iv) it is sufficient to show that $C^{(n)}(t) = \sum_{k=1}^{N^{(n)}(t)} Y_k^{(n)} = \lambda_n^{-1/2} \sum_{k=1}^{N^{(n)}(t)} Z_{n,k}$ converges to $B(t)$ in distribution, that is, $C^{(n)}(t)$ converges in distribution to a Gaussian variable $N(0,t)$ with mean 0 and variance t. The convergence

$$\lambda_n^{-1/2} \sum_{k=1}^{[\lambda_n \, t]} Z_{n,k} \xrightarrow{d} N(0,t), \quad \text{as } n \to \infty,$$

results by Lyapunov's central limit theorem ([4], Section 42), where $[\xi]$ denotes the largest integer smaller than ξ. Consider the sequence of random variables

$$W_n = \lambda_n^{-1/2} \left(\sum_{k=1}^{N_n(t)} Z_{n,k} - \sum_{k=1}^{[\lambda_n \, t]} Z_{n,k} \right), \quad n = 1, 2, \ldots,$$

and note that $E\big[W_n^2\big] = \lambda_n^{-1} \, E\big[|N_n(t) - [\lambda_n \, t]|\big] \, E\big[Z_{n,1}^2\big]$. Since $\sup_{n\geq 1} E\big[Z_{n,1}^2\big] < \infty$ by the postulated uniform integrability, we have $E\big[W_n^2\big] \to 0$ as $n \to \infty$, so that $W_n \xrightarrow{\text{pr}} 0$ by Tchebychev's inequality. The above considerations show that

$$\lambda_n^{-1/2} \sum_{k=1}^{N_n(t)} Z_{n,k} = W_n + \sum_{k=1}^{[\lambda_n \, t]} Z_{n,k}$$

converges in distribution to $N(0,1)$ as $n \to \infty$.

Consider Eqs. 9 and 10 in which $h(u) = \lambda \left(E\big[e^{i\,u\,Y_1}\big] - 1\right)$ is replaced by $\lambda_n \left(E\big[e^{i\,u\,Y_1^{(n)}}\big] - 1\right)$. The solutions φ_{X_n} of these equations are characteristic functions for each n. Arguments as in the previous section can be used to show that the limits of these versions of Eqs. 9 and 10 as $n \to \infty$ give differential equations for $\lim_{n\to\infty} \varphi_{X_n}$. For example, Eq. 10 yields

$$(-1)^{m+1} \, u \, \frac{d^{2\,m+1} \varphi_s(u)}{du^{2\,m+1}} - \frac{u^2}{2} \, \varphi_s(u) = 0 \tag{20}$$

as $n \to \infty$. If driving noise $C^{(n)}$ is replaced by $\sigma \, C^{(n)}$, then the coefficient $-u^2/2$ of φ_s in Eq. 20 becomes $-(\sigma \, u)^2/2$. In this case, $\sigma \, C^{(n)}$ converges to the scaled Brownian motion $\sigma \, B$ as $n \to \infty$.

We note that differential equations for the characteristic function of X in Eq. 2 with $V = B$ or $V = \sigma \, B$ could have been obtained directly by averaging the Itô's formula in Eq. 1 corresponding to this driving noise.

3.3 Poisson and Gaussian white noise

Suppose that $V = \sigma\, B + C$, where B is a Brownian motion process and $\sigma \in \mathbb{R}$ is a constant. The transient and stationary characteristic functions of X in Eq. 2 driven by V are the solutions of

$$\frac{\partial \varphi(u;t)}{\partial t} = (-1)^{m+1}\, u\, \frac{\partial^{2\,m+1}\varphi(u;t)}{\partial u^{2\,m+1}} - \frac{(\sigma\, u)^2}{2}\, \varphi(u;t) + \lambda \left(E\left[e^{i\,u\,Y_1}\right] - 1 \right) \varphi(u;t) \tag{21}$$

and

$$(-1)^{m+1}\, u\, \frac{d^{2\,m+1}\varphi_s(u)}{du^{2\,m+1}} - \frac{(\sigma\, u)^2}{2}\, \varphi_s(u) + \lambda \left(E\left[e^{i\,u\,Y_1}\right] - 1 \right) \varphi_s(u) = 0, \tag{22}$$

respectively. We note that the jumps of C can be any type of random variables, for example, Y_1 can be a zero mean variable in L_2 or can be an α-stable variable as defined by Eq. 16.

4 Approximate Levy white noise

Let X be the solution of Eq. 2 driven by $V = L_\alpha$ and $\alpha \in (0,2)$, as considered in the previous section. Our objective here is to develop an approximation for L_α, and a method for calculating the characteristic function of X based on this approximation.

There are notable similarities and differences between compound Poisson and α-stable processes. Let $\mathcal{M}(dt, dy)$ be a random measure defining the number of jumps of a process in $(t, t+\Delta t] \times (y, y+dy]$. The expectation of $\mathcal{M}(dt, dy)$ is $\lambda\, dt\, dF(y)$ for the compound Poisson process C in Eq. 3 and $\lambda_L(dy)\, dt$ for an α-stable process L_α, where F denotes the distribution of the jumps $\{Y_k\}$ of C,

$$\lambda_L(dy) = \frac{\alpha\, c_\alpha}{2}\, |y|^{-(\alpha+1)}\, dy, \quad y \in \mathbb{R} \setminus \{0\}, \tag{23}$$

and ([12], Property 1.2.15)

$$c_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\, \cos(\pi\, \alpha/2)}, & \text{if } \alpha \neq 1 \\ 2/\pi & \text{if } \alpha = 1. \end{cases} \tag{24}$$

Let Λ be an arbitrary Borel set in $\mathbb{R}$. The integral $(1/dt)\, \int_\Lambda \mathcal{M}(dt, dy)\, dy$ gives the average number of jumps in Λ per unit of time. This number is $\lambda \int_\Lambda dF(y) \leq \lambda < \infty$ for C but can be finite or not for L_α depending on Λ. For example, if $\Lambda = (-a, a)^c$ for some $0 < a < \infty$, then

$$\begin{aligned} (1/dt) \int_{(-a,a)^c} \mathcal{M}(dt, dy)\, dy &= 2 \int_a^\infty \lambda_L(dy) = \frac{c_\alpha}{a^\alpha} < \infty \quad \text{and} \\ (1/dt) \int_{(-a,a)} \mathcal{M}(dt, dy)\, dy &= 2 \int_{0+}^a \lambda_L(dy) = \infty, \end{aligned} \tag{25}$$

that is, the average number of jumps of L_α with magnitude larger and smaller than $a > 0$ is finite and unbounded, respectively.

4.1 Large jumps of L_α

Let $\{Y_{a,k}\}$ be the jumps of L_α with magnitude larger than $a > 0$. Consider the compound Poisson process (Eq. 3)

$$C_{\alpha,a}(t) = \sum_{k=1}^{N_a(t)} Y_{a,k} \tag{26}$$

where N_a is a Poisson counting process with intensity $\lambda_a = c_\alpha / a^\alpha$ and $\{Y_{a,k}\}$ are independent identically distributed random variables. The distribution $F_a(y)$ of $Y_{a,1}$ is the conditional probability $P\big(A_y \mid A_a\big) = P\big(A_a \cap A_y\big)/P\big(A_a\big)$, where $A_a = \{|Y_{a,1}| < a\}$ and $A_y = \{Y_{a,1} < y\}$, $y \in \mathbb{R}$, or

$$\begin{aligned} F_a(y) &= \frac{\int_{(-a,a)^c \cap (-\infty,y)} \lambda_L(dz)}{\int_{(-a,a)^c} \lambda_L(dz)} = \frac{\alpha\, a^\alpha}{2} \int_{(-a,a)^c \cap (-\infty,y)} z^{-(\alpha+1)}\, dz \\ &= \frac{\alpha\, a^\alpha}{2} \left[\frac{1}{\alpha\, |y|^\alpha}\, 1\big(y < -a\big) + \frac{1}{\alpha\, a^\alpha}\, 1\big(y \geq -a\big) + \frac{1 - (a/y)^\alpha}{\alpha\, a^\alpha}\, 1\big(y \geq a\big) \right], \quad y \in \mathbb{R}. \end{aligned} \tag{27}$$

The density and the characteristic functions of $Y_{a,1}$ are

$$f_a(y) = \frac{\alpha\, a^\alpha}{2} \left[|y|^{-(\alpha+1)}\, 1\big(y < -a\big) + y^{-(\alpha+1)}\, 1\big(y \geq a\big) \right], \quad y \in \mathbb{R} \tag{28}$$

and

$$\varphi_a(u) = \alpha\, a^\alpha \int_a^\infty \cos(u\, y)\, y^{-(\alpha+1)}\, dy, \quad u \in \mathbb{R}, \tag{29}$$

respectively. The asymptotic behavior of the distribution and density functions in Eqs. 27 and 28 as $y \to \infty$ is $y^{-\alpha}$ and $y^{-(\alpha+1)}$, respectively.

Suppose that the driving noise $V = L_\alpha$ in Eq. 2 is approximated by the compound Poisson process $C_{\alpha,a}$ in Eq. 26, and let X_a denote the corresponding diffusion. The characteristic function of X_a satisfies Eq. 9 or 10 with $(\lambda_a, Y_{a,1})$ in place of (λ, Y_1), that is, with

$$h_a(u) = \lambda_a \left(1 - E\big[e^{i\, u\, Y_{a,1}}\big] \right) \quad \text{in place of} \quad h(u) = \lambda \left(1 - E\big[e^{i\, u\, Y_1}\big] \right).$$

The function $h_a(u)$ can be calculated from

$$h_a(u) = \alpha\, c_\alpha \int_a^\infty \big(1 - \cos(u\, y)\big)\, y^{-(\alpha+1)}\, dy = \alpha\, c_\alpha\, u^\alpha \int_{u\, a}^\infty \big(1 - \cos(\xi)\big)\, \xi^{-(\alpha+1)}\, d\xi \tag{30}$$

for $u > 0$, and converges to $c_\alpha\, u^\alpha \int_0^\infty \xi^{-\alpha} \sin(\xi)\, d\xi$ as $a \to 0$. Similar calculations give $\lim_{a \to 0} h_a(u) = |u|^\alpha$.

Considerations and arguments similar to those used to write Eqs. 9 and 10 can be applied to establish differential equations for the characteristic function $\varphi; \varphi_s$ of the diffusion process X in Eq. 2 with $V = L_\alpha$. These equations result by taking the limit as $a \to 0$ of Eqs. 9 and 10 with $h_a(u)$ in place of $h(u)$, and are

$$\frac{\partial \varphi(u;t)}{\partial t} = (-1)^{m+1}\, u\, \frac{\partial^{2\,m+1} \varphi(u;t)}{\partial u^{2\,m+1}} - \varphi(u;t)\, |u|^\alpha \tag{31}$$

and

$$(-1)^{m+1}\, u\, \frac{d^{2\,m+1}\varphi_s(u)}{du^{2\,m+1}} - \varphi_s(u)\,|u|^\alpha = 0, \tag{32}$$

where $\varphi_s(u) = \lim_{t\to\infty} \varphi(u;t)$ as previously.

4.2 Small jumps of L_α

Let $L_{\alpha,a}$ be a stochastic process obtained from L_α by excluding the jumps of L_α with magnitude larger than some $a > 0$, that is, the process

$$L_{\alpha,a}(t) = L_\alpha(t) - C_{\alpha,a}(t) \tag{33}$$

with $C_{\alpha,a}$ in Eq. 26. The processes $L_{\alpha,a}$ and $C_{\alpha,a}$ have some remarkable properties. For example,

(1) $L_{\alpha,a}$ and $C_{\alpha,a}$ are independent Lévy processes ([10], Theorem 39, p. 30),

(2) $L_{\alpha,a}$ has finite absolute moments of any order since it is a Lévy process with bounded jumps ([10], Theorem 34, p. 25),

(3) $L_{\alpha,a}$ has mean 0 since its density is an even function,

(4) The variance of $L_{\alpha,a}(t)$ is linear in t since $L_{\alpha,a}(t)$ has stationary independent increments ([7], Section 3.6.4), and

(5) $L_{\alpha,a}(t)$ can be approximated by [1]

$$L_{\alpha,a}(t) \simeq \sigma(\alpha,a)\, B(t), \quad t > 0, \tag{34}$$

where B is a standard Brownian motion and

$$\sigma(\alpha,a)^2 = E\big[L_{\alpha,a}(1)^2\big] = \int_{-a}^{a} y^2\, \lambda_L(dy) = \frac{\alpha}{2-\alpha}\, c_\alpha\, a^{2-\alpha}. \tag{35}$$

The result in Eq. 34 shows that (1) a symmetric α-stable Lévy process L_α can be represented approximately by

$$L_\alpha(t) \simeq \sigma(\alpha,a)\, B(t) + C_{\alpha,a}(t), \tag{36}$$

that is, L_α can be viewed as the sum of a scaled Brownian motion $\sigma(\alpha,a)\,B$ and a compound Poisson process $C_{\alpha,a}$, that are independent of each other, and (2) the characteristic function of X in Eq. 2 with $V = L_\alpha$ can be calculated from Eqs. 21 or 22 with $\sigma(\alpha,a)$ and $Y_{\alpha,1}$ in place of σ and Y_1, respectively, that is, partial differential equations for the characteristic functions of diffusions driven by sums of Gaussian and Poisson white noise processes.

5 Numerical examples

Two example are presented to illustrate the application and accuracy of the methods in the paper for calculating the characteristic function of the class of diffusion processes defined by Eq. 2 with $V = L_\alpha$. In the first example $m = 0$, so that the characteristic function of X can be written directly since Eq. 2 has a linear drift. In the second example $m = 1$. The stationary characteristic function of X is calculated analytically from Eq. 32, and numerically by the finite difference method from Eq. 10 with $h(u)$ replaced by $h_a(u)$ in Eq. 30 and from Eq. 22 with $\sigma(\alpha, a)\, B + C_{\alpha,a}$ in Eq. 36 in place of $\sigma\, B + C$. Analytical and numerical results are compare with estimates of the characteristic function of X obtained from samples of this process.

Example 1. Let $m = 0$ and $\alpha = 1$, so that Eq. 32 becomes $\varphi_s'(u) + \varphi_s(u) = 0$ for $u \geq 0$. The solution of this equation requires only the boundary condition $\varphi_s(0) = 1$, and is $\varphi_s(u) = \exp(-u)$, $u \geq 0$. Since φ_s is an even function, we have $\varphi_s(u) = \exp(-|u|)$, $u \in \mathbb{R}$.

This result is correct since X satisfies the stochastic differential equation $dX(t) = -X(t-)\, dt + dL_\alpha(t)$, so that the random variable $X(t)$ is a symmetric α-stable variable with scale at power α equal to $\big(1 - e^{-\alpha\, t}\big)/\alpha$ ([6], Example 3.27, p. 112) and characteristic function

$$\varphi(u;t) = \exp\big[-|u|^\alpha \big(1 - e^{-\alpha\, t}\big)/\alpha\big] \to \varphi_s(u) = \exp\big(-|u|^\alpha/\alpha\big), \quad t \to \infty. \tag{37}$$

Example 2. Let $m = 1$ and $\alpha = 1$, so that Eq. 32 becomes $\varphi_s'''(u) - \varphi_s(u) = 0$ for $u \geq 0$. The general solution of this equation is

$$\varphi_s(u) = c_1\, e^u + e^{-u/2} \left(c_2 \cos\big(\sqrt{3}\, u/2\big) + c_3 \sin\big(\sqrt{3}\, |u|/2\big) \right), \quad u \geq 0, \tag{38}$$

where c_1, c_2, c_3 are some constants. The boundary conditions $\varphi_s(0) = 1$ and $|\varphi_s(u)| \leq 1$ imply $c_1 = 0$ and $c_2 = 1$. We use $|\varphi_s(u)| \leq 1$ in place of the boundary condition $\varphi_s(\infty) = 0$ proposed in [5]. The boundary condition $\varphi_s'(0) = 0$ in Eq. 13 gives $c_3 = 1/\sqrt{3}$ so that

$$\varphi_s(u) = e^{-|u|/2} \left(\cos\big(\sqrt{3}\, u/2\big) + \sin\big(\sqrt{3}\, |u|/2\big)/\sqrt{3} \right), \quad u \in \mathbb{R}. \tag{39}$$

The expression of the characteristic function of $X(t)$ in Eq. 39 coincides with that in [5].

The Fourier transform,

$$f_s(x) = \frac{1}{2\,\pi} \int_{\mathbb{R}} e^{-i\,u\,x}\, \varphi_s(u)\, du = \frac{1}{\pi} \int_0^\infty \cos\big(u\, x\big)\, \varphi_s(u)\, du, \tag{40}$$

of φ_s, that is, the stationary density f_s of X, has the expression

$$f_s(x) = \frac{1}{2\,\pi^2\, \big(a_1^2 + 1/4\big)\, \big(a_2^2 + 1/4\big)}, \quad x \in \mathbb{R}, \tag{41}$$

where $a_1 = x + \sqrt{3}/2$ and $a_2 = -x + \sqrt{3}/2$. Hence, the stationary density of X is such that $f_s(x) \sim O(x^{-4})$ as $|x| \to \infty$ in agreement with findings in [11].

Figure 1 shows with solid and dotted lines an estimate of φ_s and φ_s in Eq. 39, re-

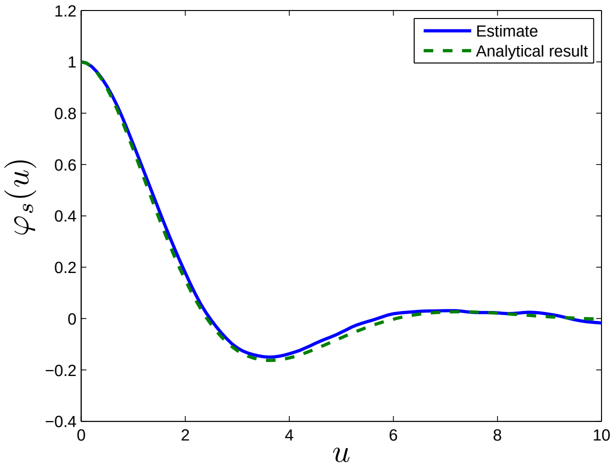

Figure 1: Estimate of φ_s and analytical expression of φ_s in Eq. 39

spectively. The estimate has been obtained from 1,000 independent samples of $X(t)$ at time $t = 10$. There is a very good agreement between the estimated and the calculated characteristic functions.

The forward finite difference approximation,

$$X(t + \Delta t) = X(t) - X(t)^3\, \Delta t + Z\, \Delta t, \tag{42}$$

has been used to generate samples of X and obtain the estimate of φ_s in Fig. 1, where $\Delta t > 0$ is a time step and Z denotes an $S_1(1, 0, 0)$ random variable. Samples of Z for $\alpha = 1$ have been generated from samples of a random variable U uniformly distributed in $(-\pi/2, \pi/2)$ and the transformation $Z = \tan(U)$. For calculation, the open interval $(-\pi/2, \pi/2)$ defining the range of U has been replaced by the closed interval $[-\pi/2 + \varepsilon, \pi/2 - \varepsilon]$ for some $\varepsilon > 0$. Relatively small values of Δt and ε need to be used in simulation to obtain reliable estimates for φ_s. Numerical experiments have shown that the estimates of φ_s are stable for values of Δt and ε equal to or smaller than 10^{-5} and 10^{-8}, respectively. Results in Fig. 1 are for $\Delta t = 5 * 10^{-6}$ and $\varepsilon = 10^{-8}$.

Figure 2 examines the accuracy of an approximation L_α consisting of the large jumps of this process, that is, the approximation $L_\alpha \simeq C_{\alpha,a}$ with $C_{\alpha,a}$ in Eq. 26. The figure shows the estimate of the characteristic function φ_s in Fig. 1 and finite difference solutions for the stationary characteristic function of X_a in Eq. 2 with $V = C_{\alpha,a}$, $\alpha = 1$, and $m = 1$. The finite difference results in Fig. 2 have been obtained from Eq. 15 with $\lambda = \lambda_a$, $E\left[e^{i\, u\, Y_1}\right] = \varphi_a(u)$, $\bar{u} = 20$, $n = 12,000$, and boundary conditions $\varphi_0 = 1$, $\varphi_1 = \varphi_{-1}$, and $\varphi_{n+k} = 0$, $k \geq 1$. The plots show that the finite difference solutions for the characteristic function φ_s (1) do not depend on a in a small vicinity of $u = 0$, (2) are indistinguishable from the estimate of φ_s around $u = 0$, (3) differ from the estimate of φ_s and among themselves away from $u = 0$ for relatively large values of a, and (4) approach the estimate of φ_s as $a \to 0$.

Figure 3 shows in a logarithmic scale the upper tails of the densities f_s obtained from the finite difference solutions of the characteristic functions φ_s in Fig. 2 by Fourier transform, and the function x^{-4} representing the asymptotic behavior of f_s (Eq. 41). The upper tail of

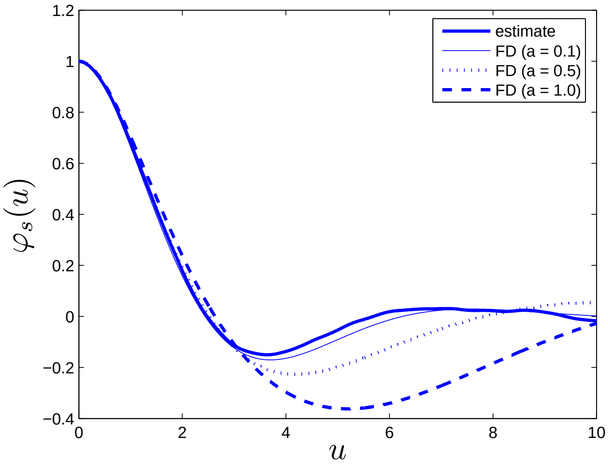

Figure 2: Estimate of φ_s (Fig. 1) and finite difference solutions of φ_s for diffusions driven by $V = C_{\alpha,a}$ with $\alpha = 1$ and $a = 0.1$, 0.5, and 1

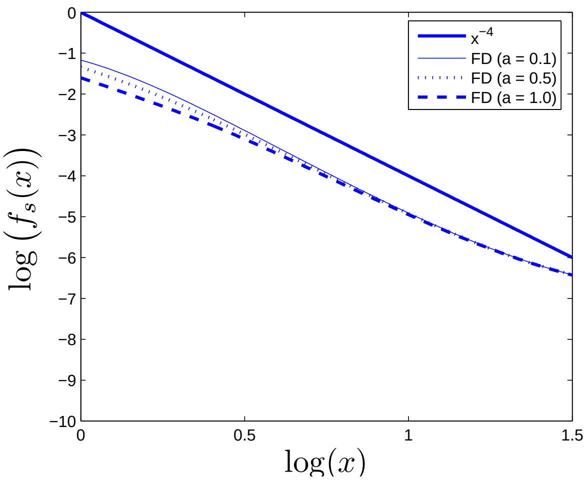

Figure 3: Upper tail of f_s corresponding to the finite difference solutions of the characteristic functions φ_s in Fig. 2 and function x^{-4}

the density f_s depends weakly on a and exhibits approximately the x^{-4} power law behavior. The convergence of the upper tails of the density of X_a is consistent with a theoretical result in [11].

Figure 4 examines the accuracy of an approximation of L_α consisting of the large jumps of this process superposed on a scaled Brownian motion, that is, the approximation $L_\alpha \simeq \sigma(\alpha, a)\, B + C_{\alpha,a}$ in Eq. 36. The figure shows the characteristic functions for diffusions defined by Eq. 2 with $V = C_{\alpha,a}$ and $V = \sigma(\alpha, a)\, B + C_{\alpha,a}$ and $a = 1$, and the estimate of the characteristic function of X in Eq. 2 with $V = L_\alpha$. The agreement between the characteristic functions of diffusions under $\sigma(\alpha, a)\, B + C_{\alpha,a}$ and L_α is impressive. The scale $\sigma(\alpha, a)$ of the Brownian motion has been obtained from Eq. 35 with $\alpha = 1$, $a = 1$, and c_α in Eq. 24, and is $\sigma(1, 1) = \sqrt{2/\pi}$.

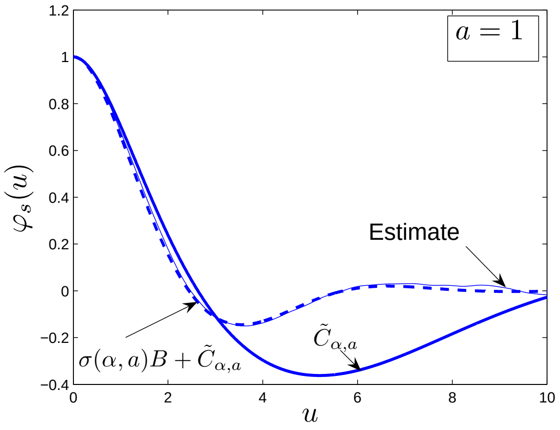

Figure 4: Estimate of φ_s (Fig. 1) and finite difference solutions of φ_s for diffusions driven by $C_{\alpha,a}$ and $\sigma(\alpha, a)\, B + C_{\alpha,a}$ with $a = 1$, $\alpha = 1$, and $\sigma(\alpha, a) = \sqrt{2/\pi}$

6 Conclusions

A practical method has been developed for calculating the characteristic function of diffusions driven by a class of Lévy white noise processes defined as the formal derivative of symmetric α-stable Lévy motion processes L_α. The method is based on (1) Itô's formula for semimartingales, (2) a differential equation developed for the characteristic function of diffusions driven by sums of compound Poisson and Brownian motion processes, and (3) approximate representations of L_α consisting of the large jumps of L_α or the large jumps of L_α superposed on scaled Brownian motions. Numerical results show that the proposed method is very accurate and is consistent with previous theoretical findings [11].

A notable result in the paper is the development of a differential equation for the characteristic function of diffusions driven by compound Poisson processes with jumps that may not have finite moments. The result has been used to derive partial differential equations for the characteristic function of diffusions driven by Gaussian and Lévy white noise processes as limits of partial differential equations for diffusions driven by Poisson white noise. These equations and approximations of symmetric α-stable Lévy motion processes have been used to develop a practical method for analyzing diffusions driven by Lévy white noise. The differential equation for the characteristic function of diffusions driven by Lévy white noise obtained in the paper coincides that reported in [5].

7 Appendix 1. Existence of density

We show that the stationary version of the diffusion process X in Eq. 2 with the compound Poisson driving noise $V = C$ in Eq. 3 has a density. Our argument does not apply to diffusions driven by general α-stable Lévy motion processes. The existence of the density of solutions of Eq. 2 driven by α-stable Lévy motion processes with $0 < \alpha < 1$ at any positive time can be derived by using appropriate truncation techniques and appealing to Theorem 1.2 in [3]. We conjecture that the stationary solution has a density for all

$0 < \alpha < 2$.

The existence of the density for the stationary solution in the compound Poisson driven case results from the following two claims. As previously stated, $(X(t),\, t \geq 0)$ is a stationary process satisfying Eq. 2 with $V = C$ in Eq. 3.

Claim 1. $X(0)$ has a density in $\mathbb{R} \setminus \{0\}$. We need to show that for every $\varepsilon > 0$, $X(0)$ has a density on $\{x : |x| \geq \varepsilon\}$. For (notational) simplicity we consider positive x and $\varepsilon = 1$. The existence of a density will follow once we prove that there is a finite constant $c > 0$ such that for any interval $(a, b]$ with $1 \leq a < b$ we have

$$P\big(a < X(t) \leq b\big) \leq C(b - a)\,. \tag{43}$$

Denote $p = P\big(X(0) \in (a, b]\big)$. Suppose that the last jump of X occurred before $T = 0$ and X has no jump till a time $t > 0$. Then (Eq. 11)

$$|X(t)| = \Big(|X(0)|^{-2\,m} + 2\,m\,t\Big)^{-1/(2\,m)}, \tag{44}$$

so that

$$|X(t_1)| = \Big(|X(0)|^{-2\,m} + a^{-2\,m} - b^{-2\,m}\Big)^{-1/(2\,m)}, \tag{45}$$

at time $t_1 = \big(a^{-2\,m} - b^{-2\,m}\big)/(2\,m)$ provided X has no jump in $(0, t_1)$. If $X(0) \in (a, b]$, we have $X(t_1) \in (a_1, a)$, $a_1 = \big(2\,a^{-2\,m} - b^{-2\,m}\big)^{-1/(2\,m)}$, and by stationarity

$$P\big(X(0) \in (a_1, a]\big) = P\big(X(t_1) \in (a_1, a]\big) \geq p\, e^{-\lambda\, t_1}\,. \tag{46}$$

The intervals $(a_1, a]$ and $(a, b]$ are disjoint.

Suppose now that $X(0) \in (a_1, a]$ and consider a time $t_2 = \big(a_1^{-2\,m} - a^{-2\,m}\big)/(2\,m) = t_1$. Then $X(t_2) = X(t_1)$ takes values in $(a_2, a_1]$ with $a_2 = \big(2\,a_1^{-2\,m} - a^{-2\,m}\big)^{-1/(2\,m)}$. The intervals $(a_2, a_1]$ and $(a_1, a]$ are disjoint and, by stationarity and Eq. 46, we have

$$P\big(X(0) \in (a_2, a_1)\big) = P\big(X(t_1) \in (a_2, a_1)\big) \geq P\big(X(0) \in (a_1, a)\big)\, e^{-\lambda\, t_1} \geq p\, e^{-2\,\lambda\, t_1}. \tag{47}$$

We proceed in the same manner, constructing disjoint intervals $I_j = (a_j, b_j]$ for $j = 0, 1, 2, \ldots$ and $a_0 = a$, $b_0 = b$, with $P(X(0) \in I_j) \geq p\, e^{-j\,\lambda\, t_1}$ for all j. Since the intervals I_j are disjoint, we must have $P\big(\cup_{j=1}^{\infty} \{X(0) \in I_j\}\big) \leq 1$, so that

$$1 \geq \sum_{j=0}^{\infty} P\big(X(0) \in I_j\big) \geq \sum_{j=0}^{\infty} p\, e^{-j\,\lambda\, t_1} = p\, \frac{1}{1 - e^{-\lambda\, t_1}}. \tag{48}$$

and

$$p = P\big(X(0) \in (a, b]\big) \leq \big(1 - e^{-\lambda\, t_1}\big) \leq \lambda\, t_1 \leq \lambda\,(b - a)\,, \tag{49}$$

that is, the claimed property for $P\big(X(0) \in (a, b]\big)$. To establish the final upper bound on p we used the inequality $a^{-2\,m} - b^{-2\,m} \leq 2\,m\,(b - a)$.

Claim 2. A stationary solution $X(0)$ has no atom at 0. Let T_1 denotes the first jump of C. The probability of the event $\{X(t)=0\}$ for some $t \geq 0$ is

$$\begin{aligned} P\big(X(t)=0\big) &= P\big(X(t)=0, T_1 > t\big) + P\big(X(t)=0, T_1 \leq t\big) \\ &\leq P\big(T_1 > t\big) + P\big(X(t)=0, T_1 \leq t\big) = e^{-\lambda t} + P\big(X(t)=0, T_1 \leq t\big). \end{aligned} \tag{50}$$

We show

$$P\big(X(t)=0 \text{ for some } t \geq T_1\big) = 0, \tag{51}$$

so that $P\big(X(t)=0\big) \leq e^{-\lambda t}$ implying $P\big(X(t)=0\big) \to 0$ as $t \to \infty$. Since $P\big(X(t)=0\big) = P\big(X(0)=0\big)$ for every $t \geq 0$ by stationarity, this will imply the claim.

That Eq. 51 holds follows from two observations. First, by Eq. 44, the process $(X(t))$ cannot hit zero between the times of the jumps of the driving noise. Second, at the times of the jumps of the driving noise, we have a recurrence formula (Eq. 11)

$$|X(T_j)| = \big(|X(T_{j-1})|^{-2\,m} + 2\,m\,E_j\big)^{-1/(2\,m)} + \Delta C(T_j) \tag{52}$$

where $\{T_j\}$ denote the jump times of C, $\{E_j = T_j - T_{j-1}\}$ are independent exponential variables with mean $1/\lambda$, and $\Delta C(T_j)$ is the jump of C at time T_j. The continuity of the exponential random variable E_j in the formula in Eq. 52 shows that the event $\{X(T_j)=0\}$ has probability 0.

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